# On the disturbed motion of a plane vortex sheet

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SUMMARY

A formal solution to the initial value problem for a plane vortex sheet in an inviscid fluid is obtained by transform methods. The eigenvalue problem is investigated and the stability criterion determined. This criterion is found to be in agreement with that obtained previously by Landau (1944), Hatanaka (1949), and Pai (1954), all of whom had included spurious eigenvalues in their analyses. It is also established that supersonic disturbances may be unstable; related investigations in hydrodynamic stability have conjectured on this possibility, but the vortex sheet appears to afford the first definite example. Finally, an asymptotic approximation is developed for the displacement of a vortex sheet following a suddenly imposed, spatially periodic velocity.

## 1. INTRODUCTION

The instability of a plane vortex sheet has been investigated for incompressible flow in the classical work of Helmholtz, Rayleigh, and Kelvin (see Lamb 1945, § 232, § 268) and for compressible flow by Landau (1944), Hatanaka (1949)* and Pai (1954). Each of these last three investigations ignored the existence of branch points for the eigenvalue equation and accepted the eigenvalues given by its two possible branches. We shall establish the proper treatment of the branch points and the ultimate character of the motion by considering an initial value problem for the vortex sheet and obtaining an asymptotic solution for large time. Our results confirm those of Landau, Hatanaka, and Pai with respect to the question of stability but rule out certain of their neutral eigenvalues.

A second feature of the vortex sheet problem that appears to have received insufficient emphasis is the fact that supersonic disturbances, i.e. disturbances that have a supersonic wave speed relative to the local flow, can be unstable. Such disturbances have often been neglected in other stability problems on the (sometimes tacit, sometimes conjectured) assumption that they could not be unstable, so that it is especially important to ascertain that their presence in the results of Landau, Hatanaka, and Pai is not a consequence of their failure to deal properly with the aforementioned branch points. We also remark that neutral supersonic disturbances of a vortex sheet are significant for acoustic reflection therefrom (Miles 1957).

* I am indebted to Professor I. Imai for a résumé of Hatanaka's paper, which is not readily available in the United States.

A more direct derivation of the eigenvalue equation could be achieved by posing a simple, travelling wave disturbance (see (4.11) below). The proper restrictions on the eigenvalue equation (and, therefore, the proper conclusions regarding the question of stability) could then be inferred from Sommerfeld's finiteness and radiation conditions*, although it is to be noted that the propriety of the radiation condition in a stability investigation does not appear to have been accepted unequivocally (see Lin's remarks quoted in § 6 below, especially his implied question as to the nature of a 'proper restriction')†. Aside from this question, however, the possibilities for even asymptotic solutions of initial value problems in hydrodynamic stability appear quite limited, and the vortex sheet is probably one of the few such problems that is tractable.

## 2. Formulation of the problem

We consider (see figure 1) two ideal fluids occupying the half spaces $y > 0$ and $y < 0$, designated by the subscripts $+$ and $-$, respectively, and each characterized by its uniform velocity $U_{\pm}$ parallel to the $x$-axis, sonic velocity $a_{\pm}$, and density $\rho_{\pm}$. The vortex sheet separating the two fluids is subjected to an initial displacement $n_0(x)$ and an initial velocity $\dot{n}_0(x)$; we wish to examine its subsequent motion and, in particular, the conditions under which this motion will be bounded (i.e. stable) for large time.

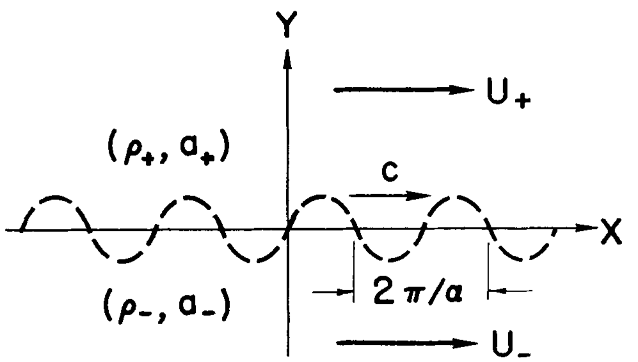


Figure 1. Vortex sheet separating two parallel flows.

Many of the following equations involve the parameters of only one fluid or the other; accordingly, we need include the subscripts $\pm$ only in those equations that involve the parameters of both fluids, with the implication that equations devoid of the $\pm$ subscripts apply to either fluid.

* I am indebted to Professor G. Carrier for persistently refusing to accept certain erroneous conclusions that were improperly inferred from the radiation condition in the original manuscript. Extensive discussions with Professor Carrier then led to the attack on the initial value problem.

† Stoker (1952, p. 97) has considered an initial value, surface wave problem in order to illustrate the validity of the radiation condition in forced oscillation problems, but his results do not appear to be directly applicable to eigenvalue problems.

Assuming small disturbances, the perturbation pressure satisfies the wave equation

$$a^2\nabla^2 p = \frac{D^2 p}{Dt^2}, \tag{2.1}$$

where

$$\frac{D}{Dt} = \frac{\partial}{\partial t} + U\frac{\partial}{\partial x}. \tag{2.2}$$

The boundary conditions at the vortex sheet $y = n(x, t)$, across which the pressure must be continuous and to which the fluid motion must be tangential, are

$$p_+ = p_-, \qquad -\frac{1}{\rho}\frac{\partial p}{\partial y} = \frac{D^2 n}{Dt^2}, \qquad y \to 0\pm, \tag{2.3}$$

while the boundary conditions at infinity are

$$\lim_{|x|\to\infty} |p| < \infty, \qquad \lim_{|y|\to\infty} |p| < \infty. \tag{2.4}$$

The initial conditions are

$$p = \frac{\partial p}{\partial t} = 0, \qquad t \leqslant 0, \qquad y \neq 0 \tag{2.5}$$

and

$$n = n_0(x), \qquad \frac{\partial n}{\partial t} = \dot{n}_0(x), \qquad t = 0. \tag{2.6}$$

We require solutions to (2.1) in $y < 0$ and $y > 0$ that satisfy equations (2.3) to (2.6).

## 3. The transform solution

We define $P$, the Fourier–Laplace transform of $p$, by

$$P(y; m, s) = \int_0^\infty e^{-st}\, dt \int_{-\infty}^\infty e^{-imx} p(x, y, t)\, dx, \tag{3.1 a}$$

$$p(x, y, t) = \frac{1}{4\pi^2 i}\int_{-\infty}^\infty e^{imx}\, dm \int_{\gamma-i\infty}^{\gamma+i\infty} e^{st} P(y; m, s)\, ds, \tag{3.1 b}$$

where $\gamma$ is a positive real number such that all singularities of $P$ lie in $\mathscr{R}\{s\} < \gamma$. Similarly, we define $N(m, s)$ as the Fourier–Laplace transform of $n(x, t)$ and

$$N_0(m) = \int_{-\infty}^\infty e^{-imx} n_0(x)\, dx, \tag{3.2 a}$$

$$V_0(m) = \int_{-\infty}^\infty e^{-imx}\left[\dot{n}_0(x) + U\frac{dn_0(x)}{dx}\right] dx, \tag{3.2 b}$$

as the Fourier transforms of the initial displacement and transverse velocities imparted to the fluids. We also introduce the operator

$$S = s + imU, \tag{3.3}$$

corresponding to the substantial time derivative of (2.2). The transform of the wave equation (2.1), subject to the initial conditions (2.5), then reads

$$a^2\left(\frac{\partial^2 P}{\partial y^2} - m^2 P\right) = S^2 P, \tag{3.4}$$

while the transforms of the boundary conditions (2.3), subject to the initial conditions (2.6), read

$$P_+ = P_-, \qquad -\frac{1}{\rho}\frac{\partial P}{\partial y} = S^2N - SN_0 - V_0. \tag{3.5}$$

Solving (3.4) subject to (3.5) at $y = 0$, we obtain

$$P_\pm(y;\, m, s) = \frac{im\rho_+\rho_-(U_+ - U_-)(S_+V_{0-} + S_-V_{0+})e^{-\mu_\pm|y|}}{\mu_+\mu_-(Q_+ + Q_-)} \tag{3.6}$$

and
$$N(m,s) = \frac{(\rho_+/\mu_+)(S_+N_0 + V_{0+}) + (\rho_-/\mu_-)(S_-N_0 + V_{0-})}{Q_+ + Q_-}, \tag{3.7}$$

where

$$Q = \rho S^2/\mu \tag{3.8}$$

and

$$\mu = \{m^2 + (S/a)^2\}^{1/2}, \qquad \mathscr{R}\{\mu\} \geqslant 0. \tag{3.9}$$

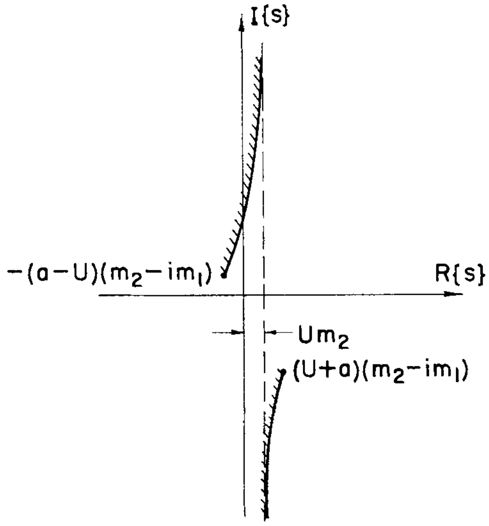


Figure 2. Cuts for $\mu(s, m)$ in the $s$-plane for $U < a$, $m_1 > 0$, $m_2 > 0$.

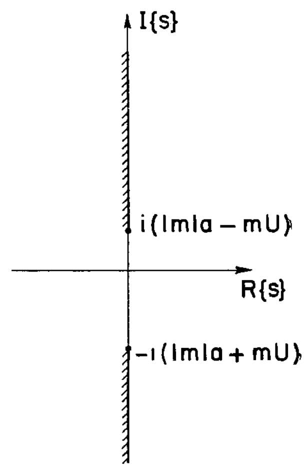


Figure 3. Cuts for $\mu(s, m)$ in the $s$-plane for $m$ real.

The condition $\mathscr{R}\{\mu\} \geqslant 0$, which is a consequence of the second finiteness condition (2.4), will be satisfied everywhere in the complex $s$- and $m$-planes if we choose the branch cuts from the branch points

$$s + im(U \pm a) = 0 \tag{3.10}$$

to be lines on which $\mathscr{R}\{\mu\} = 0$ or, equivalently, $\mu^2 \leqslant 0$; the latter condition yields

$$a^2 m_1 m_2 + (s_1 - Um_2)(s_2 + Um_1) = 0, \tag{3.11 a}$$

$$a^2(m_1^2 - m_2^2) + (s_1 - Um_2)^2 - (s_2 + Um_1)^2 \leqslant 0, \tag{3.11 b}$$

where

$$m = m_1 + im_2, \qquad s = s_1 + is_2. \tag{3.12}$$

Equations (3.11 a, b) define the cuts as segments of a hyperbola in the $m$-plane for fixed $s$, or in the $s$-plane for fixed $m$; if $m_2 = 0$ these segments degenerate into straight lines in the $s$-plane, and similarly in the $m$-plane if $s_2 = 0$. The cuts in the $s$-plane for $U < a$, $m_1 > 0$, and $m_2 > 0$ are illustrated in figure 2; if $m_1 < 0$ the cuts from $(U+a)(m_2 - im_1)$ and $-(a-U)(m_2 - im_1)$ must be asymptotic to $Um_2 + i\infty$ and $Um_2 - i\infty$. The cuts for $m_2 = 0$, as in the problem to be considered in the following section, are shown in figure 3.

To be sure, the condition $\mathscr{R}\{\mu_\pm\} \geqslant 0$ need not be satisfied everywhere in the $s$- and $m$-planes but only along the paths of integration for the inverse transforms of $P_\pm$ when $\pm y > 0$. We imply here only that it is sufficient to choose the cuts defined by (3.11 a, b); in the final analyses they may be deformed in any manner that ensures convergence of the integrals and the satisfaction of the second finiteness condition (2.4) by $p_\pm$.

## 4. Spatially periodic disturbances

It suffices, for the question of stability, to consider a spatially periodic (fixed $m$) disturbance. Such a disturbance will result if we assume both $n_0$ and $\dot{n}_0$ to be periodic in $x$, but the algebra is simplest and the results equally illuminating for the special case

$$n_0(x) = 0, \qquad \dot{n}_0(x) = v_0 e^{i\alpha x}, \tag{4.1}$$

where $v_0$ is the amplitude of an initial, transverse velocity imparted to the vortex sheet at $t = 0$ and $\alpha$ the wave-number (figure 1). We must take $\alpha$ to be real in order to satisfy (2.4), and we may assume it to be positive without loss of generality. The required transforms of (4.1) are

$$N_0 = 0, \qquad V_0 = 2\pi v_0 \delta(m-\alpha), \tag{4.2}$$

where $\delta(m-\alpha)$ denotes the Dirac delta function. Substituting (4.2) in (3.6) and (3.7), writing the inverse transforms according to (3.1 b) and carrying out the $m$-integrations yields

$$p_\pm(x,y,t) = \frac{\alpha\rho_+\rho_-(U_+-U_-)v_0}{2\pi}\int_{\gamma-i\infty}^{\gamma+i\infty}\frac{(S_+ + S_-)e^{st+i\alpha x-\mu_\pm|y|}\,ds}{\mu_+\mu_-(Q_+ + Q_-)} \tag{4.3}$$

and

$$n(x,t) = \frac{v_0}{2\pi i}\int_{\gamma-i\infty}^{\gamma+i\infty}\left[\frac{(\rho_+/\mu_+)+(\rho_-/\mu_-)}{Q_+ + Q_-}\right]e^{st+i\alpha x}\,ds, \tag{4.4}$$

where $m = \alpha$ in $\mu$, $S$, and $Q$.

It is found convenient, especially in relating our results to more conventional, travelling wave analyses, to introduce the change of variable

$$s = -i\alpha c, \qquad \gamma = \alpha\epsilon, \tag{4.5}$$

where $c$ is a complex wave speed. Writing also

$$\beta = \frac{i\mu}{\alpha} = \left[\left(\frac{c-U}{a}\right)^2 - 1\right]^{1/2}, \qquad \mathscr{I}\{\beta\} \geqslant 0, \tag{4.6}$$

and

$$F = iQ/\alpha = \rho(c-U)^2/\beta, \tag{4.7}$$

(4.3) and (4.4) become

$$p_{\pm}(x,y,t) = \frac{\rho_+\rho_-(U_+-U_-)v_0}{2\pi i}\int_{-\infty+i\epsilon}^{\infty+i\epsilon}\frac{(2c-U_+-U_-)e^{i\alpha(x-ct+\beta_\pm|y|)}\,dc}{\beta_+\beta_-(F_++F_-)} \tag{4.8}$$

and

$$n(x,t) = -\frac{v_0}{2\pi\alpha}\int_{-\infty+i\epsilon}^{\infty+i\epsilon}\left[\frac{(\rho_+/\beta_+)+(\rho_-/\beta_-)}{F_++F_-}\right]e^{i\alpha(x-ct)}\,dc. \tag{4.9}$$

The branch points and cuts for $\beta$ are shown in figure 4, as also are its phase angles on the axes $\mathscr{I}\{c\} = 0$ and $\mathscr{R}\{c\} = U$ (the latter is significant for an observer moving with the fluid, i.e. in the coordinates $x-Ut$ and $y$).

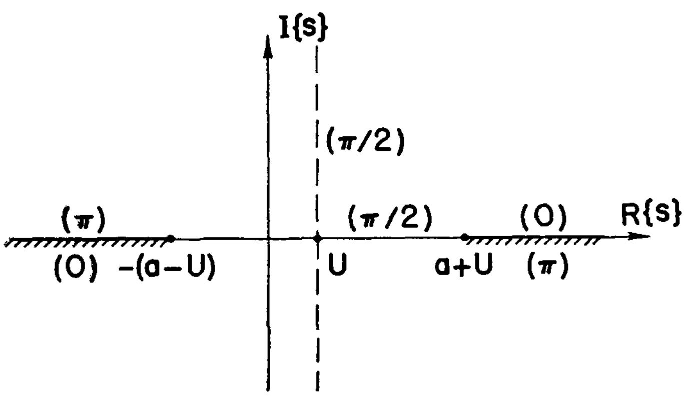


Figure 4. Cuts for $\beta$ in $c$-plane for $m$ real; the phases of $\beta$ on the real axis and on $\mathscr{R}\{c\} = U$ are indicated in parentheses.

There are a total of four branch points, namely, $c = U_\pm + a_\pm$, $U_\pm - a_\pm$, for the integrands of (4.8) and (4.9). We may assume $U_+ > U_-$ without loss of generality (if $U_- > U_+$ we have only to replace $y$ by $-y$), and there are then four possible configurations of the branch points, corresponding to (see figure 5)

$A_1$: $0 < U_+ - U_- < a_- - a_+$,

$A_2$: $0 < U_+ - U_- < a_+ - a_-$,

$A_3$: $|a_+ - a_-| < U_+ - U_- < a_+ + a_-$,

B: $a_+ + a_- < U_+ - U_-$.

The three possibilities $A_{1,2,3}$ are not significantly different (in particular, as shown in § 5, the essential character of the eigenvalues is the same for all), but in B we find that it is impossible for a disturbance to be subsonic, i.e. $|\mathscr{R}\{c\} - U| < a$, with respect to both fluids. We also note that the cuts for $\beta_\mp$ may be deformed freely in evaluating $p_\pm$, as given by (4.8); moreover, only two finite segments of the real axis appeai as cuts for the integrand of (4.9), the phase jump of the quantity in square brackets vanishing if $\beta_+$ and $\beta_-$ are either both real or both imaginary.

We now consider the nature of the elementary, exponential disturbances from which (4.8) is synthesized. This may be done conveniently in the coordinates $x-Ut$ and $y$, especially if we introduce the transformation

$$c = U+a\sec\theta, \qquad \beta = \tan\theta, \tag{4.10}$$

under which

$$\exp\{i\alpha(x-ct+\beta|y|)\} = \exp\{i\alpha\sec\theta[(x-Ut)\cos\theta+|y|\sin\theta-at]\}. \tag{4.11}$$

If $\theta$ is real, as it is for the points on cuts in the $c$-plane, this corresponds to an outgoing (incoming) plane wave of sound in $x-Ut$ and $y$ for

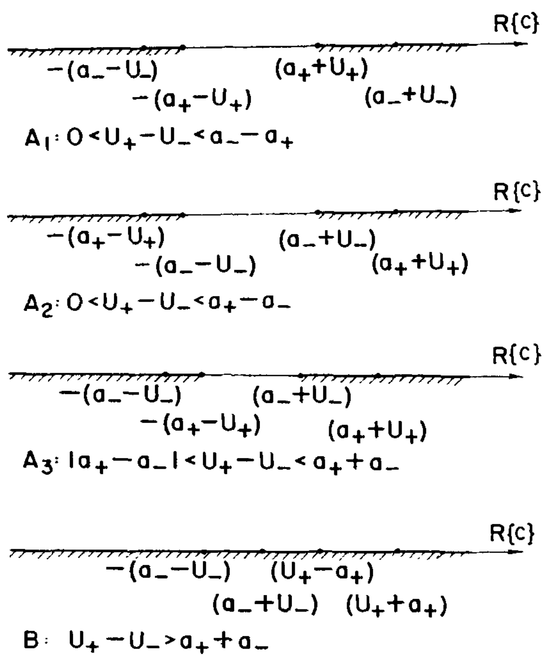


Figure 5. Cuts for $\beta_+$ and $\beta_-$ in $c$-plane for $m$ real.

$\sin\theta>(<)\,0$; more generally, points in $\mathscr{I}\{c\}>(<)\,0$, correspond to outgoing (incoming) waves that fall off exponentially as $|y|\to\infty$. Of course, the total disturbance is confined to $|y|<at$, for if $|y|>at$ we may close the path of integration in $\mathscr{I}\{c\}>\epsilon$ to obtain

$$p(x,y,t)\equiv 0, \qquad |y|>at. \tag{4.12}$$

It follows that the envelope of the total disturbance is the *outgoing* wave-front $|y|=at$, as is otherwise directly evident from the definition of $a$.

## 5. The eigenvalue problem

The vortex sheet at $y=0$ is stable only if the integrand of (4.9) has no poles in $\mathscr{I}\{c\}>0$; otherwise it is unstable. These poles correspond to the zeros of

$$F_+(c)+F_-(c) = \rho_+(c-U_+)^2/\beta_+ + \rho_-(c-U_-)^2/\beta_- \tag{5.1 a}$$

$$= \rho_+a_+^2(\beta_++\beta_+^{-1})+\rho_-a_-^2(\beta_-+\beta_-^{-1}) \tag{5.1 b}$$

and may be identified as the eigenvalues for elementary disturbances of the type (4.11) subject to the boundary conditions (2.3) and (2.4). We remark that (5.1) does not contain the wave-number and is therefore significant for any disturbance that has the phase velocity $c$.

We may simplify the algebra considerably, without unduly restricting the results, by assuming equal specific heat ratios in the two fluids. Then, in virtue of the requirement of equal static pressures,

$$\rho_+ a_+^2 = \rho_- a_-^2, \tag{5.2}$$

and we may factor (5.1 b) to obtain*

$$(\beta_+ + \beta_-)(\beta_+ \beta_- + 1) = 0. \tag{5.3 a}$$

Alternatively, the trigonometric substitution (4.10) yields

$$\sin(\theta_+ + \theta_-)\cos(\theta_+ - \theta_-) = 0. \tag{5.3 b}$$

We consider first the zeros of $\beta_+ + \beta_-$. Remarking that these zeros must be real in consequence of the finiteness condition $\mathscr{I}\{\beta\} \geqslant 0$, and that $\beta_+$ and $\beta_-$ can be of opposite sign only on the real axis interval $U_- + a_- < c < U_+ - a_+$ (which arises only for $U_+ - U_- > a_+ + a_-$), we deduce that

$$\beta_+ + \beta_- = 0 \quad \text{at } c = \left(\frac{a_+ U_- + a_- U_+}{a_+ + a_-}\right), \qquad U_+ - U_- > a_+ + a_-. \tag{5.4}$$

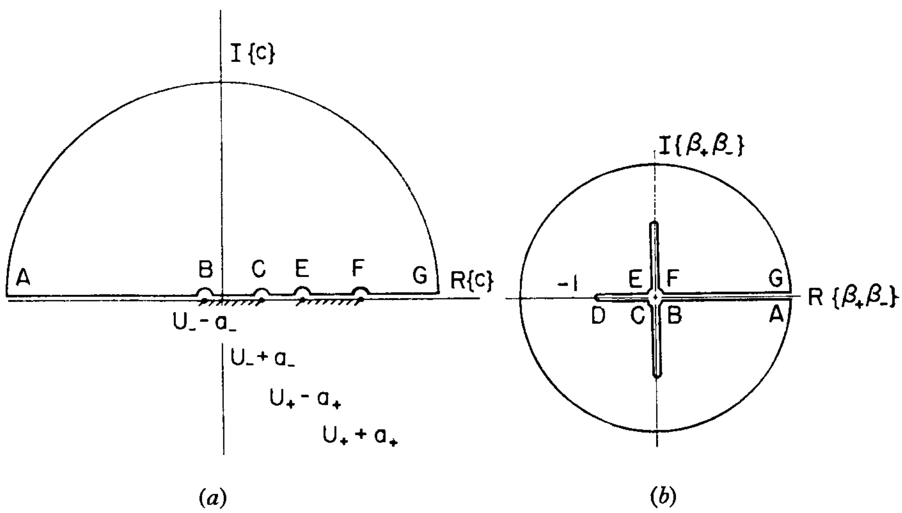


Figure 6. Mapping of the upper half $c$-plane on the $\beta_+\beta_-$-plane.

This zero corresponds to $\theta_+ + \theta_- = \pi$ in (5.3 b). It clearly represents a neutral, supersonic disturbance and was obtained by Landau and Pai along with the spurious, real zero of $\beta_+ - \beta_- = 0$.

We may investigate the possible zeros of $\beta_+ \beta_- + 1$ (equivalent to those of $\cos(\theta_+ - \theta_-)$) in $\mathscr{I}\{c\} \geqslant 0$ by applying Cauchy's principle of the argument

* The eigenvalue equation considered by Landau and Pai was

$$(\beta_+ \beta_-^2 - 1)(\beta_+^2 - \beta_-^2) = 0,$$

while Hatanaka considered $(\beta_+^2 \beta_-^2 - 1) = 0$.

to the contour consisting of the real axis of the $c$-plane, indented above the branch points of $\beta_+$ and $\beta_-$, and a semi-circle of radius tending to infinity in the upper half-plane. This contour and its map on a $\beta_+\beta_-$-plane (the Cauchy–Nyquist diagram) are shown in figures 6 (*a*), (*b*) on the assumption $U_+ - U_- > a_+ + a_-$; the general shape of the $\beta_+\beta_-$ contour is similar for the other branch point configurations of figure 5, only the order in which the points are traversed changing. It is evident that the $\beta_+\beta_-$ contour will encircle the point $-1$ once if $(\beta_+\beta_-)_D > -1$ (as shown in figure 6 (*b*)) or will pass through $-1$ twice if $(\beta_+\beta_-)_D < -1$, from which we infer that $\beta_+\beta_- + 1$ must have either one zero in $\mathscr{I}\{c\} > 0$ or two zeros on the real $c$-axis, respectively. Similarly, we infer from a contour indented below the branch points and closed in the lower half of the $c$-plane that $\beta_+\beta_- + 1$ may have one zero in $\mathscr{I}\{c\} < 0$ if $(\beta_+\beta_-)_D > -1$. We also remark that both of the $\beta_+\beta_-$ contours pass through the point $+1$ twice (independently of the location of the point $(\beta_+\beta_-)_D$), corresponding to the two real zeros of $\beta_+\beta_- - 1$; these cannot be equal, being less than $U_- - a_-$ and greater than $U_+ + a_+$, respectively*.

The critical point $(\beta_+\beta_-)_D = -1$ may be determined by requiring $\beta_+\beta_- + 1$ and its derivative with respect to $c$ to vanish simultaneously; now, however, having proved that $\beta_+\beta_- - 1$ cannot have a double root, we may work with $\beta_+^2\beta_-^2 - 1$ or, more conveniently,

$$1 - \left(\frac{a_+}{c - U_+}\right)^2 - \left(\frac{a_-}{c - U_-}\right)^2 = 0, \tag{5.5}$$

which is readily shown to have the double root

$$c = \left(\frac{a_+^{2/3} U_- + a_-^{2/3} U_+}{a_+^{2/3} + a_-^{2/3}}\right) \tag{5.6 a}$$

at

$$(U_+ - U_-)^{2/3} = a_+^{2/3} + a_-^{2/3}. \tag{5.6 b}$$

We conclude, therefore, that

$$(\beta_+\beta_-)_D \gtrless -1 \quad \text{as } (U_+ - U_-) \lessgtr (a_+^{2/3} + a_-^{2/3})^{3/2}. \tag{5.7}$$

The limiting case $U_+ = U_-$ ($= U$, say) requires special consideration, since the two complex conjugate zeros of $\beta_+\beta_- + 1$ then coalesce at $c = U$. We infer that an infinitesimal, tangential discontinuity in a compressible fluid will exhibit *linear* (rather than *exponential*) instability, thereby generalizing Rayleigh's result (Lamb 1945, § 232) for incompressible flow.

The foregoing results are summarized in table 1, and we conclude that, subject to the restriction $\rho_+ a_+^2 = \rho_- a_-^2$, a necessary and sufficient condition for the stability of an inviscid vortex sheet with respect to small disturbances is

$$|U_+ - U_-| > (a_+^{2/3} + a_-^{2/3})^{3/2}, \tag{5.8}$$

* These last zeros were included in the analyses of Landau, Hatanaka, and Pai; indeed, the chief flaw in these analyses, insofar as they aim only at a stability criterion, is their failure to prove that the complex zeros of $\beta_+^2\beta_-^2 - 1$ may be charged to $\beta_+\beta_- + 1$ alone.

in agreement with the predictions of Landau (1944), Hatanaka (1949), and Pai (1954)*.

| Relative speed range | Zeros of (5.3) | | |
|---|---|---|---|
| | $\mathscr{I}\{c\} = 0$ | $\mathscr{I}\{c\} > 0$ (unstable) | $\mathscr{I}\{c\} < 0$ |
| $0 = \|U_+ - U_-\|$ | 2* | 0 | 0 |
| $0 < \|U_+ - U_-\| < a_+ + a_-$ | 0 | 1 | 1 |
| $a_+ + a_- < \|U_+ - U_-\| < (a_+^{2/3} + a_-^{2/3})^{3/2}$ | 1 | 1 | 1 |
| $(a_+^{2/3} + a_-^{2/3})^{3/2} < \|U_+ - U_-\|$ | 3 | 0 | 0 |

Table 1. The possible eigenvalues of equation (5.3).

* double root at $c = U$.

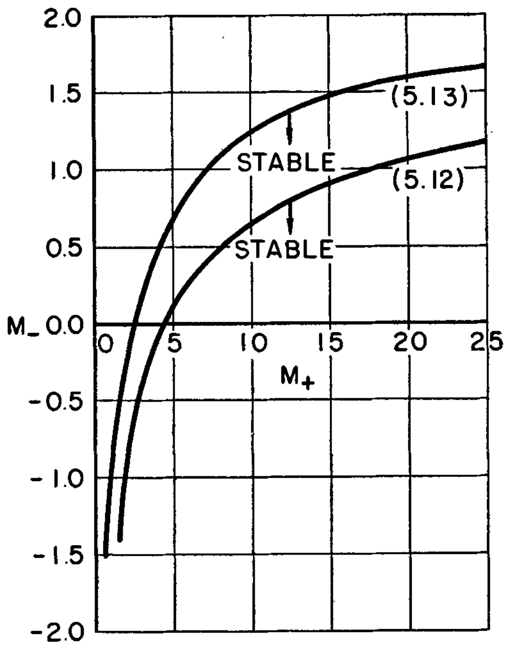


Figure 7. The stability boundaries of (5.12) and (5.13).

The explicit determination of the zeros of $\beta_+\beta_- + 1$ requires the solution of a quartic equation, but in the special case of equal sonic velocities ($a_+ = a_- = a$) this may be reduced to a quadratic equation having the roots

$$c = \tfrac{1}{2}(U_+ + U_-) \pm \tfrac{1}{2}a[M^2 + 4 - 4(M^2+1)^{1/2}]^{1/2}, \quad M = (U_+ \mp U_-)/a, \quad (5.9)$$

* Neither Hatanaka nor Pai gave the result (5.8) in explicit form, but their final results are in agreement therewith.

where $M$ is the Mach number of relative flow, which, according to (5.7), must exceed $2^{3/2}$ for stability.

A special case of greater practical significance than that of equal sonic velocities is that of equal stagnation enthalpies (e.g. the vortex sheets in shock interaction patterns) where

$$\tfrac{1}{2}U_+^2 + \frac{a_+^2}{\gamma-1} = \tfrac{1}{2}U_-^2 + \frac{a_-^2}{\gamma-1}. \tag{5.10}$$

Introducing the notation

$$M_\pm = U_\pm/a_\pm, \qquad m = a_+/a_-, \tag{5.11}$$

and combining (5.10) with (5.6 b), we obtain the parametric representation

$$M_- = (\gamma-1)^{-1}(1+m^{2/3})^{-3/2}(1-m^2) - \tfrac{1}{2}(1+m^{2/3})^{3/2}, \tag{5.12 a}$$

$$M_+ = (\gamma-1)^{-1}m^{-1}(1+m^{2/3})^{-3/2}(1-m^2) + \tfrac{1}{2}m^{-1}(1+m^{2/3})^{3/2}. \tag{5.12 b}$$

This stability boundary is plotted in figure 7 for $\gamma = 1{\cdot}4$, as also is that obtained by combining (5.10) with $U_+ - U_- = a_+ + a_-$ (see Lin (1953) and § 6 below), namely

$$M_- = \tfrac{1}{2}[(3-\gamma)-(\gamma+1)m]/(\gamma-1), \tag{5.13 a}$$

$$M_+ = \tfrac{1}{2}[-(3-\gamma)+(\gamma+1)m^{-1}]/(\gamma-1). \tag{5.13 b}$$

## 6. The role of supersonic disturbances

Supersonic disturbances, defined as those for which $|\mathscr{R}\{c\} - U| > a$, have played a somewhat ambiguous role in previous investigations of hydrodynamic stability*. Referring to these disturbances for a *boundary layer*, Lin (1955, pp. 70, 71) states that:

"One would then expect the disturbance outside the boundary layer to be a wave with non-diminishing amplitude at infinity instead of an exponential decay of the amplitude. There is no discrete characteristic value problem for such disturbances unless some proper restriction is imposed. In fact, these 'supersonic disturbances' have not yet been fully studied.

In all the existing theoretical analyses of the stability of the boundary layer in a gas, supersonic disturbances are assumed to be insignificant. This is based on the conjecture that the energy associated with such disturbances would propagate from the boundary layer in the nature of acoustic waves. Additional theoretical work and experimental evidence on this point are highly desirable."

If we had presumed the impossibility of unstable supersonic disturbances for our model of a vortex sheet, the condition

$$|U_+ - U_-| > a_+ + a_- \tag{6.1}$$

* If $|\mathscr{R}\{c\} - U| > a$ the elementary disturbance of (4.11) has a supersonic phase velocity, $\mathscr{R}\{c\} - U$, along the coordinate axis $x - Ut$. The phase velocity with respect to the fluid is $[1+(\mathscr{R}\{\beta\})^2]^{-1/2}(\mathscr{R}\{c\} - U)$; if $c$ and $\beta$ are both real, the latter velocity is equal to the sonic velocity $a$, as is directly evident from (4.11).

would have appeared *a priori* as *sufficient* for stability. The more severe criterion (5.8) is a consequence of the existence of unstable supersonic disturbances for

$$a_+ + a_- < |U_+ - U_-| < (a_+^{2/3} + a_-^{2/3})^{3/2}.$$

This, in our opinion, implies the possibility of their existence in related problems of hydrodynamic stability and adds emphasis to the concluding sentence in the above quotation from Lin. In particular, we should expect (5.8), which constitutes a necessary condition for the stability of a mixing region of zero thickness, to be significant for the stability of a laminar mixing region of finite thickness with respect to inviscid disturbances having wavelengths large compared with the thickness. The latter problem has been studied by Lin, who suggested the criterion (6.1) as sufficient for stability on the tentative hypothesis that only subsonic disturbances could be unstable. It is not clear that the analysis presented here is directly comparable with Lin's*, but the present results certainly enhance the desirability of its extension to include supersonic disturbances.

We remark that the presence of a boundary at some finite value of $y$, such as the wall in the boundary layer problem or the virtual boundaries for symmetric and antisymmetric disturbances of a symmetric jet (cf. Pai 1951), may be decisive for the stability of supersonic disturbances. A vortex sheet parallel to a boundary at which either $p_y = 0$ or $p = 0$ is considered in the Appendix and is found not to admit *neutral*, supersonic disturbances; it does not necessarily follow that unstable, supersonic disturbances are impossible for this configuration, but it seems likely that this also should be so.

## 7. Asymptotic evaluation of $n$ for $a_- = a_+$

We now return to the solution obtained in § 4. An explicit evaluation of the integrals (4.8) and (4.9) in terms of known functions does not appear to be feasible, but an asymptotic development for $n$ is straightforward. We may further simplify this development, without losing any significant features, by assuming

$$\rho_+ = \rho_- = \rho, \qquad a_+ = a_- = a, \tag{7.1}$$

so that (4.9) reduces to

$$n(x,t) = -\frac{v_0}{2\pi\alpha a^2}\int_{-\infty+i\epsilon}^{\infty+i\epsilon} \frac{e^{i\alpha(x-ct)}\,dc}{\beta_+\beta_-+1}. \tag{7.2}$$

* In applying boundary conditions, Lin assumes that the displacement of any streamline must be small compared with the thickness of the mixing region, so that the approximations appear rather different than those adopted here. It is possible that the approximation of small displacement could be relaxed to one of small streamline slope simply by an implicit change of variable, e.g. a von Mises transformation in which $y$ is replaced by the stream function. A second feature of the finite mixing region that is absent in the model of a vortex sheet is the *inner viscous region* (Lin 1955, p. 136), where $c = U$ and the inviscid differential equation has a singularity. The role of this singularity as the thickness of the mixing region tends to zero would require special attention.

The symmetry of this special case may be emphasized by shifting to a coordinate system $(x_0, y)$ that moves with the mean velocity $c_0$ of the two fluids, where

$$x_0 = x - c_0 t, \qquad c_0 = \tfrac{1}{2}(U_+ + U_-). \tag{7.3}$$

We also introduce the dimensionless wave speed $w$, such that

$$c = c_0 + aw, \qquad \beta_\pm = \{(w \mp \tfrac{1}{2}M)^2 - 1\}^{1/2}, \qquad \epsilon = a\epsilon', \tag{7.4}$$

where $M$ is defined by (5.9); (7.2) then becomes

$$n(x,t) = -\frac{v_0 e^{i\alpha x_0}}{2\pi\alpha a}\int_{-\infty+i\epsilon'}^{\infty+i\epsilon'} \frac{e^{-i\alpha a t w}\,dw}{\beta_+\beta_- + 1}. \tag{7.5}$$

The poles of the integrand are given by (5.9) as

$$w = \pm i\lambda, \qquad \lambda = [(M^2+1)^{1/2} - (\tfrac{1}{4}M^2+1)]^{1/2}. \tag{7.6}$$

The complex $w$-plane, the original path of integration $C$, and the cuts and poles of $\beta_+\beta_- + 1$ are shown in figure 8; if $M > 2^{3/2}$ the poles lie on the real axis between $\pm|\tfrac{1}{2}M - 1|$, rather than on the imaginary axis.

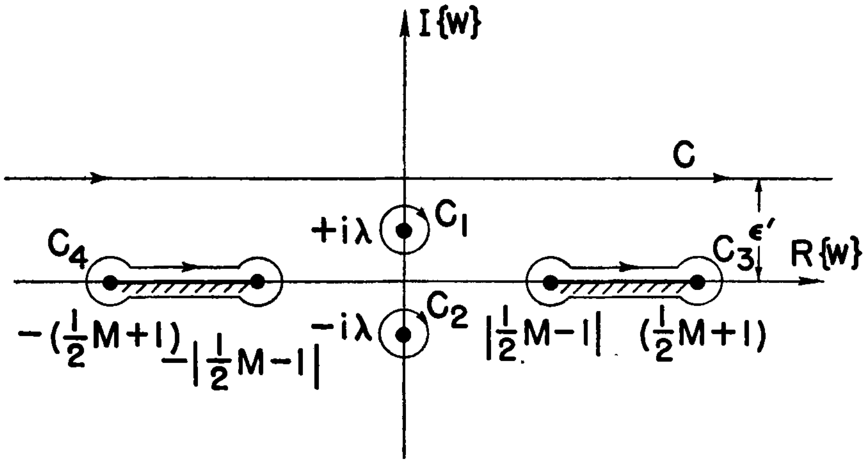


Figure 8. The $w$-plane for (7.5).

If $t > 0$ we may deform $C$ into the four, clockwise loops $C_{1,2,3,4}$ shown in figure 8, plus a contour at infinity in the lower half-plane that makes a null contribution to the result. The phases of $\beta_+\beta_-$ on the top and bottom of $C_3$ are $\tfrac{1}{2}\pi$ and $\tfrac{3}{2}\pi$ and conversely for $C_4$; hence, the integrals over the top of $C_3$ and the bottom of $C_4$ are found to differ only in the sign of the exponent, and similarly for those over the bottom of $C_3$ and the top of $C_4$. Utilizing these data to combine the four integrals over the tops and bottoms of the cuts and evaluating the contributions of the poles with the aid of Cauchy's residue theorem yields

$$n(x,t) = \frac{v_0 e^{i\alpha x_0}}{\alpha a}\left\{\frac{\sinh(\alpha\lambda a t)}{\lambda(M^2+1)^{1/2}} + \right.$$
$$\left. + \frac{2}{\pi}\mathscr{I}\int_{|\frac{1}{2}M-1|}^{\frac{1}{2}M+1} \frac{\sin(\alpha a t w)\,dw}{1 - i\{w^2 - (\tfrac{1}{2}M-1)^2\}^{1/2}\{(\tfrac{1}{2}M+1)^2 - w^2\}^{1/2}}\right. . \tag{7.7}$$

If $M > 2^{3/2}$,

$$\sinh(\alpha\lambda at)/\lambda = \sin(\alpha|\lambda|at)/|\lambda|. \tag{7.8}$$

The imaginary part of the integrand of (7.7) vanishes like $|w - |\frac{1}{2}M+1||^{1/2}$ at the end points and has no other singularities or points of stationary phase. Accordingly, integrating once by parts and making use of 2·8 (11) in Erdélyi's (1956) monograph, we obtain the asymptotic approximation

$$n(x,t) = \frac{v_0\, e^{i\alpha(x-c_0 t)}}{\alpha a}\left\{\frac{\sinh(\alpha\lambda at)}{\lambda(M^2+1)^{1/2}} \mp \right.$$

$$\left. \mp\left[\frac{2M|M\pm 2|}{\pi}\right]^{1/2}(\alpha at)^{-3/2}\cos\left[(\tfrac{1}{2}M\pm 1)\alpha at \mp \frac{\pi}{4}\right] + O[(\alpha at)^{-5/2}]\right\}, \tag{7.9}$$

where the sum of the two terms corresponding to the upper and lower choices of sign is to be taken.

We remark that a similar development is possible for the pressures, but then (assuming the cuts as in figure 5) the integral will extend along the entire real axis, and there will be contributions from points of stationary phase; moreover, if $M > 2$ the contributions of the poles on the real axis (one at $w = 0$ if $2 < M < 2^{3/2}$ or three at $w = 0$ and $\pm|\lambda|$ if $M > 2^{3/2}$) must first be separated out.

## Appendix

*Vortex sheet near boundary*

We remarked in § 6 that the presence of a boundary at some finite value of $y$ may be of decisive importance with respect to the stability of supersonic disturbances. We attempt to throw further light on this point for the vortex sheet problem by placing a wall at $y = -h$, so that

$$U = U_-, \qquad -h \leqslant y < 0, \tag{A 1a}$$

$$\phantom{U} = U_+, \qquad y > 0. \tag{A 1b}$$

We then have, in addition to the boundary conditions (2.3),

$$p_y = 0, \qquad y = -h. \tag{A 2}$$

Assuming the elementary disturbance of (4.11) and imposing (2.3) and (A 2), the resulting eigenvalue equation is found to be (cf. (5.1a))

$$\frac{\rho_+(c-U_+)^2}{\beta_+} + \frac{\rho_-(c-U_-)^2}{\beta_-}\coth(-i\alpha\beta_- h) = 0. \tag{A 3}$$

If the boundary condition $p_y = 0$ is replaced by $p = 0$ at $y = -h$, it is necessary only to replace coth by tanh in (A 3).

We remark that: (i) $\beta_-^{-1}\coth(-i\alpha\beta_- h)$ is a single-valued function of $c$, so that the only branch points of (A 3) are at $c = U_+ \pm a_+$; (ii) (A 3) reduces to (5.1a) as $h \to \infty$ unless $\beta_-$ is real (supersonic, neutral disturbance in $y < 0$), so that (A 2) differs from a radiation condition as $h \to \infty$ only if $\beta_-$ is real; (iii) there can be no supersonic, neutral disturbances in the upper medium, since $\beta_+$ is real for such disturbances, while the second term in (A 3) is imaginary for all real $c$; (iv) the statements (i) to (iii) remain valid if the hyperbolic cotangent is replaced by the hyperbolic tangent.

We *surmise* from the foregoing remarks that supersonic, neutral disturbances *may* not exist for a boundary layer near a fixed wall or for a symmetric jet (for either symmetric or antisymmetric disturbances).

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